Curved noncommutative torus and Gauss–Bonnet

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Abstract

We study perturbations of the flat geometry of the noncommutative two-dimensional torus $\mathbb{T}^2_\theta$ (with irrational θ). They are described by spectral triples $(A_\theta, \mathcal{H}, D)$, with the Dirac operator D, which is a differential operator with coefficients in the commutant of the (smooth) algebra A_θ of $\mathbb{T}_\theta$. We show, up to the second order in perturbation, that the ζ-function at 0 vanishes and so the Gauss-Bonnet theorem holds. We also calculate first two terms of the perturbative expansion of the corresponding local scalar curvature.

1 Introduction

Starting with the seminal paper [9] (see also [2]) there has been a growing interest in study of the metric and curvature issues on the most popular example in the realm of noncommutative geometry: the noncommutative two-torus $\mathbb{T}_\theta$ [13], [1], [8], [14]. This has been pursued in the framework of spectral triples [5, 6], mainly with their twisted (or modular) version, with the metric obtained from the flat one by a conformal rescaling factor.

Building on the analysis of spectral triples on the principal $U(1)$ bundles [10], [11] we propose another, more general class of curved perturbations of the flat geometry. They are described by spectral triples $(A_\theta, \mathcal{H}, D)$, with the Dirac operator D, which is a differential operator with coefficients in $M_2(\mathbb{C}) \otimes (JA_\theta J^*)$, where A_θ is the (smooth) algebra of $\mathbb{T}_\theta$ and J is the real structure (charge conjugation). The fact that $JA_\theta J^*$ is in the commutant of A_θ is here essential to guarantee that D has bounded commutators with functions in A_θ.

We take the perturbations of the standard equivariant Dirac operator on the noncommutative torus, which correspond to arbitrary perturbations of the standard flat metric. We compute that the ζ-function at 0 vanishes up to the second order in the perturbation parameter ε. Thus we show the Gauss-Bonnet formula to hold (in the same approximation) for a more general class of Dirac operators than previously studied, which strongly indicates that it should hold exactly. In section 3.4 we compute up to the order ε^2 the corresponding local scalar curvature.

We also briefly touch upon few other properties as the first order differential calculus, the orientation postulate (which holds under commutativity of certain components of the orthonormal coframe) and absolute continuity.

2 The curved geometry of torus

We start with the recollection of the Riemannian geometry of the classical 2-dimensional torus.

2.1 Classical torus $\mathbb{T}^2$

Let $x^\mu \in [0, 2\pi)$, $\mu = 1, 2$ be the coordinates on $\mathbb{T}^2 = S^1 \times S^1$. The derivations $\delta_\mu = i\frac{\partial}{\partial x^\mu}$ generate the usual action of $U(1)\times U(1)$ on $\mathbb{T}^2$ and on its algebra of smooth functions $A = C^\infty(\mathbb{T}^2, \mathbb{C})$. Given a metric tensor $g = g_{\mu\nu}dx^\mu dx^\nu$ on $\mathbb{T}^2$ we introduce and use a global orthonormal frame (basis) $e = \{e_j\}$, $e_j = \{e^\mu_j \partial_\mu\}$, $j = 1, 2$, of the tangent bundle $T\mathbb{T}^2$. Clearly $e' = \{e'_j\}$ is another orthonormal frame for g if and only if e' differs from e by a (point dependent) rotation. Note that the 2×2 matrix of smooth real valued functions e^μ_j is nondegenerate at all points of $\mathbb{T}^2$ and that one has

$$\sum_j e^\mu_j e^\nu_j = g^{\mu\nu},$$

where $g^{\mu\nu}$ is the inverse matrix of $g_{\mu\nu}$. The metric gives rise in a natural way to a measure on the torus $\mu_g = \sqrt{|g|}\mu_0$, where $|g| = \det(g_{\mu\nu})$
and by μ_0 we denote the standard Lebesgue measure on $\mathbb{T}^2$ (corresponding to the flat metric).
It is straightforward to see that the scalar curvature has the following expression in terms of the orthonormal frame:

$$R = 2\,\mathcal{L}_{e_i}(c_{ijj}) - c_{kii}c_{kjj} - \frac{1}{4}c_{ijk}c_{ijk} - \frac{1}{2}c_{ijk}c_{kji}\,, \tag{2.1}$$

where c_{ijk} are the structure constants of the commutators (of vector fields)

$$[e_i, e_j] = c_{ijk}e_k.$$

To make contact with our computations in the noncommutative case, we present here the perturbative expansion of the scalar curvature R. Assuming the small perturbation of the globally flat orthonormal frame:

$$e_i = e^\mu_i \partial_\mu, \qquad e^\mu_i = \delta^\mu_i + \varepsilon h^\mu_i,$$

where h_{ij} is a matrix of arbitrary smooth real functions on the manifold, we obtain (up to ε^2)

$$\begin{aligned} R \;&= 2\varepsilon\left(h^i_{i,jj} - h^j_{i,ij}\right) + 2\varepsilon^2 h^j_i\left(2h^k_{k,ij} - h^k_{i,jk} - h^i_{j,kk}\right) \\ &+ \frac{1}{2}\varepsilon^2 h^j_{i,k}\left(h^j_{k,i} - h^j_{i,k} + 3h^k_{j,i} - h^k_{i,j} - 5h^i_{j,k}\right) \\ &+ \varepsilon^2 h^i_{i,j}\left(2h^k_{j,k} + 2h^j_{k,k} - h^k_{k,j}\right) - \varepsilon^2 h^j_{i,j}h^k_{i,k}. \end{aligned} \tag{2.2}$$

We shall also need the perturbative expansion of $\sqrt{|g|}\,R$, which since

$$\sqrt{|g|} \sim 1 - \varepsilon(h^k_k),$$

has just the terms of order ε^2

$$-\,2\varepsilon^2 h^k_k\left((h^i_i)_{jj} - (h^j_i)_{ij}\right), \tag{2.3}$$

in addition to the terms in the expansion (2.2) of the scalar curvature R alone.
On $\mathbb{T}^2$ there are four spin structures but for any of them we can take the associated spinor bundle $\Sigma\mathbb{T}^2$ as the trivial rank 2 hermitian vector bundle on $\mathbb{T}^2$. A $U(1)$ action either lifts directly to a spin structure and then to an action

$$\kappa : U(1) \times \Sigma\mathbb{T}^2 \to \Sigma\mathbb{T}^2,$$

and so it lifts to the action on sections $C^\infty(\mathbb{T}^2, \Sigma\mathbb{T}^2)$ of $\Sigma\mathbb{T}^2$, or to a projective action, i.e. to the action of a non-trivial double cover of $U(1)$ (which happens to be still $U(1)$ as a group). The generators of the action of $U(1) \times U(1)$, that implement (via a commutator) the two derivations δ_μ of A will be denoted by the same symbol δ_μ.
The standard Dirac operator D_g, which comes from the metric compactible and torsion-free spin connection can be globally expressed on smooth sections of $\Sigma\mathbb{T}^2$ in terms of the orthonormal frame as:

$$D_g = \sum_{j,\mu=1}^{2} \left(\sigma^j e_j^\mu \delta_\mu\right) + \frac{1}{2} c_{122}\, \sigma^1 + \frac{1}{2} c_{211}\, \sigma^2 \,, \tag{2.4}$$

where σ^j are the hermitian Pauli matrices in $M_2(\mathbb{C})$:

$$\sigma^1\sigma^2 = -\sigma^2\sigma^1 = i\sigma^3, \quad \sigma^3 := \mathrm{diag}(1, -1). \tag{2.5}$$

D_g extends to an unbounded selfadjoint operator on $L^2(\Sigma\mathbb{T}^2, \mu_g)$. To study the spectral properties we shall employ a convenient isometry of Hilbert spaces:

$$Q : L^2(\Sigma\mathbb{T}^2, \mu_g) \to L^2(\Sigma\mathbb{T}^2, \mu_0), \qquad Q(\psi) = \mathrm{det}(g)^{\frac{1}{4}}\psi,$$

and consider the operator QD_gQ^{-1} on $L^2(\Sigma\mathbb{T}^2, \mu_0)$. Note that the principal symbols of QD_gQ^{-1} and D_g are equal and only the zero-order terms are different. As the zero-order term is not significant for the computations of the curvature we can, in fact, restrict ourselves to yet another operator on $\Sigma\mathbb{T}^2$:

$$D = \sum_{j,\mu=1}^{2} \left(\sigma^j e_j^\mu \delta_\mu + \frac{1}{2}\sigma^j(\delta_\mu e_j^\mu)\right), \tag{2.6}$$

which extends to selfadjoint operator on $L^2(\Sigma\mathbb{T}^2, \mu_0)$ and differs from $\Xi D_g \Xi^{-1}$ only by a zero-order term.
Such a term in two dimensions consists of a Clifford image of some selfadjoint one-form. For the purpose of the paper the crucial fact is that $(C^\infty(\mathbb{T}^2), L^2(\Sigma T^2, \mu_0), D)$ is an even spectral triple, with the grading $\chi := \sigma^3$. Though it is not necessarily real, it contains a first order, elliptic Dirac-type operator D, which is a fluctuaction of the genuine Dirac operator, that belongs to a real spectral triple (c.f. [6]).
Moreover, the distance defined by D,

$$d_D(x, y) = \sup\{|a(x) - a(y)| \;:\; a \in A,\ \|[D, a]\| \leq 1\} \tag{2.7}$$

is exactly the geodesic distance of the metric g. This is so, because only the principal symbol of D has nontrivial commutators with $a \in C^\infty(\mathbb{T}^2)$ and the norm is still the supremum norm.
Clearly, since D is in general different from the canonical Dirac operator, it might (a priori) not minimize the Einstein action functional or (in the special case of two-dimensional torus) do not satisfy the Gauss-Bonnet theorem, stated in terms of its spectral invariant. However, as the perturbations of the Dirac by one-forms do not contribute the leading terms of the heat-kernel coefficients we notice that

the relevant spectral invariants related to curvature shall be unchanged.

Consider now two functionals which to any element $a \in C^\infty(\mathbb{T}^2)$ assign the value of the zeta function at 0

$$\zeta_{a,D}(0) = \text{Tr}\, a|D|^{-s}|_{s=0} \quad \text{and} \quad \zeta_{a,D_g}(0) = \text{Tr}\, a|D_g|^{-s}|_{s=0},$$

where Tr is the trace over the relevant Hilbert space. Since both these functionals are spectral and depend only on the principal symbol of the operators D and D_g respectively, they are necessarily identical, $\zeta_{a,D}(0) = \zeta_{a,D_g}(0)$. Assuming that both operators have empty kernel, the known expression for D_g in terms of scalar curvature, $\zeta_{a,D_g}(0) = \frac{1}{12\pi}\int_{\mathbb{T}^2} aR\,\mu_g$ permits us to have a similar expression for D

$$\zeta_{a,D}(0)\frac{1}{12\pi}\int_{\mathbb{T}^2} a\sqrt{|g|}R\,\mu_0\ ,$$

in which however we had to use the standard Lebesgue measure μ_0 that corresponds to the canonical trace on the algebra $C^\infty(\mathbb{T}^2)$. This means that from the functional $\zeta_{a,D}(0)$ we retrieve the information about the product $\sqrt{|g|}\,R$ rather than about R itself (in fact it may be not possible to define $\sqrt{|g|}$ in the noncommutative setup).

2.2 Noncommutative torus $\mathbb{T}^2_\theta$

The C^*-algebra of the 2-dimensional noncommutative torus $\mathbb{T}_\theta$ is generated by two unitary elements U_i, $i = 1, 2$, with the relations

$$U_1U_2 = e^{2\pi i\theta}U_2U_1,$$

where $0 < \theta < 1$ is irrational. The smooth subalgebra A_θ consists of all elements of the form

$$a = \sum_{k,l\in\mathbb{Z}} \alpha_{kl}U_1^kU_2^l,$$

where $\alpha_{k,l}$ is a rapidly decreasing sequence.
The natural action of $U(1) \times U(1)$ by automorphisms, gives, in its infinitesimal form, two linearly independent derivations on the algebra A_θ, given on the generators as:

$$\delta_1(U_1) = U_1, \delta_1(U_2) = 0, \quad \delta_2(U_2) = U_2, \delta_2(U_1) = 0\ . \tag{2.8}$$

The canonical trace on A_θ

$$\mathfrak{t}(a) = \alpha_{00},$$

is invariant with respect to the action of $U(1) \times U(1)$, and therefore:

$$\mathfrak{t}(\delta_j(a)) = 0, \quad \forall j = 1, 2.$$

It is easy to see that the trace extends uniquely to the C^*-algebra of the noncommutative torus.
Let $\mathcal{H}_0$ be the Hilbert space of the GNS construction with respect to the trace $\mathfrak{t}$ on $\mathbb{T}_\theta$, and π the associated faithful representation. With the orthonormal basis, $\epsilon_{k,l}$, of $\mathcal{H}_0$ we have:

$$U_1\epsilon_{k,l} = \epsilon_{k+1,l},$$
$$U_2\epsilon_{k,l} = e^{2\pi ik\theta}\epsilon_{k,l+1},$$

where k, l are in $\mathbb{Z}$ or $\mathbb{Z} + \frac{1}{2}$ depending on the choice of the spin structure (see [17]). We double the Hilbert space taking $\mathcal{H} = \mathcal{H}_0 \otimes \mathbb{C}^2$, with the diagonal representation of the algebra. We take J, the real structure to be

$$J = i\sigma^2 \circ J_0,$$

where J_0 is the canonical Tomita-Takesaki antilinear map on the Hilbert space $\mathcal{H}_0$:

$$J_0 \epsilon_{k,l} = \epsilon_{-k,-l}.$$

By construction, the conjugation by J maps the algebra A_θ to its commutant:

$$[a, Jb^* J^{-1}] = 0, \quad \forall a, b \in A_\theta.$$

The explicit action of the derivations on the basis is given by

$$\delta_1 \epsilon_{k,l} = k\epsilon_{k,l}, \quad \delta_2 \epsilon_{k,l} = l\epsilon_{k,l}.$$

Before we continue let us observe two important facts. The derivations δ_i, $i = 1, 2$ anticommute with J and hence could be extended as derivation to $(A_\theta)'$. In fact, since they are realized in the representation as commutators, they extend as derivations to the image of $A_\theta \otimes (A_\theta)'$.
Moreover, the canonical trace $\mathfrak{t}$ is also invariant under conjugation by J in the following sense:

$$\mathfrak{t}(JaJ^{-1}) = \overline{\mathfrak{t}(a)},$$

and therefore it makes sense to consider its extension to the image of $A_\theta \otimes (A_\theta)'$ by setting:

$$\mathfrak{t}(ab^o) = a_{0,0} b_{0,0},$$

where

$$a = \sum_{m,n} a_{m,n} U^m V^m, \qquad b = \sum_{m,n} b_{m,n} \tilde{U}^m \tilde{V}^n,$$

and

$$\tilde{U} = JU^* J^{-1}, \tilde{V} = JV^* J^{-1}.$$

Clearly, the functional is well-defined on the image of tensor product of smooth algebras A_θ and A_θ^o. In fact, it a trace obtained from the „flat" Laplacian on the noncommutative torus, $\Delta = (\delta_1)^2 + (\delta_2)^2$ through the formula:

$$\mathfrak{t}(ab^o) = \mathrm{Res}_{s=2} \left(\mathrm{Tr}\, \pi(a)\pi(b^0)\Delta^{-s/2} \right).$$

It follows directly from Theorem 2.6 in [12] that the trace factorizes.
We take as the Dirac-type operator D, the operator of the form:

$$D = \sum_{j,\mu=1}^{2} \left(\sigma^j e_j^\mu \delta_\mu + \frac{1}{2} \sigma^j \delta_\mu (e_j^\mu) \right), \tag{2.9}$$

where σ^i are the usual selfadjoint Pauli matrices and e_j^i are selfadjoint elements of $JA_\theta J^*$ such that the matrix e_i^j is invertible.

Such operator is of the general type of generalized differential operators on the noncommutative torus (or, more precisely, on the module over A_θ), which were introduced in studied in [3, 5]:

$$P = \sum_{\alpha,\beta} C_{\alpha,\beta}\delta_1^\alpha \delta_2^\beta,$$

where $C_{\alpha\beta}$ are A_θ endomorphisms. Clearly this is the case of D given by (2.9), as the elements of $JA_\theta J^*$ are A_θ-endomorphisms of the the trivial module.
Notice that from the assumption of invertibility of the matrix e_i^j it follows that $\sigma^i e_i^j \xi_j$ is invertible as an element of $M_2(JA_\theta J^*)$ for arbitrary $\xi \in \mathbb{R}^2 \setminus \{0\}$.

Thus, following [5][Definition 1, p.358] D could be seen as an elliptic operator.

The principal symbol of D^2 is of the form

$$\sum_{i,j} \left(P_{ij}\xi^i\xi^j\right), \tag{2.10}$$

where P_{ij} are elements of $M_2(JA_\theta J^*)$. For the operators of that type, under some additional assumptions, we can show that we recover the class of operators with compact resolvent, as one would like to have for genuine Laplace operators. We will not dwell here on this point making the conditions precise, but just sketch the argumentation. First, consider the operator

$$T = \left(P_{11}(\delta_1)^2 + P_{12}\delta_1\delta_2 + P_{22}(\delta_2)^2 + \lambda\right)(1+\triangle)^{-1},$$

where $\triangle$ is the „flat" Laplace operator on the noncommutative torus:

$$\triangle = (\delta_1)^2(\delta_2)^2,$$

and P_{ij} are elements from $JA_\theta J^*$ (or the matrix algebra over it). Note that each of the components of the sum is bounded, hence P is bounded. Further, if the operator $P = P_{ij}\delta_i\delta_j$ is assumed to be elliptic, then using similar arguments as in [4]
one can show that for some λ, $P+\lambda$ has no kernel and therefore T is invertible. Hence, one can see that $(P+\lambda)^{-1}$ is compact.
We postpone to future work the detailed analysis (which would necessarily make use of noncommutative Sobolev spaces, see [18] and [19] for first examples).

In addition to that property (which certainly holds only for some class of the operators of the type considered) we can establish few more algebraic properties. The first one regards the first order differential calculi.

Lemma 2.1. *For any D the bimodule of one forms $\Omega^1_D(A_\theta)$ is isomorphic to $A_\theta \oplus A_\theta$.*

Proof. Oberve that the forms:

$$\omega_i = (U_i)^*[D, U_i],$$

are central (that is $[a, \omega_i] = 0$ for every $a \in A_\theta$) and generate $\Omega^1(A_\theta)$ as a left (or right) module.
We only need to check that that module is free. Assume that there exist $a_i \in A_\theta$ such that $\sum_i a_i\omega_i = 0$. This implies $\sum_i e_j^i a_i = 0$ for $j = 1, 2$. However, since we assumed that matrix e_j^i was invertible, we immediately have $a_i = 0$. $\square$

The second one is the orientation property (we refer to [6] for the terminology).

Lemma 2.2. *Let $JA_\theta J^* \otimes A_\theta$ be a A_θ-bimodule with $a(JbJ^* \otimes c)d = JbJ^* \otimes acd$. For each D, there exists $c \in JA_\theta J^* \otimes A_\theta \otimes A_\theta \otimes A_\theta$,*

$$c = \frac{1}{2i} \sum_{a,b,j,k} \epsilon_{ba} E^b_k E^a_j \otimes U^*_k U^*_j \otimes U_j \otimes U_k, \tag{2.11}$$

where ϵ_{ba} is the antisymmetric tensor and E^b_k is the inverse of the matrix e^k_j, i.e. $E^b_k e^k_j = \delta^b_j$, such that

$$\pi_D(c) := \frac{1}{2i} \sum_{a,b,j,k,p,r} \epsilon_{ba} E^b_k E^a_j U^*_k U^*_j [D, U_j][D, U_k] = \chi \, ,$$

where χ is the grading operator on $\mathcal{H}$. If we view c as a Hochschild 2-chain with values in a A_θ-bimodule $JA_\theta J^* \otimes A_\theta$, where $a(JbJ^* \otimes c)d = JbJ^* \otimes bcd$, c is a cycle if

$$[E^1_1, E^2_2] = 0 \quad \text{and} \quad [E^2_1, E^1_2] = 0. \tag{2.12}$$

Proof. A straightforward calculation shows that

$$\pi_D(c) = \frac{1}{2i} \sum_{a,b,j,k,p,r} \epsilon_{ba} E^b_k E^a_j e^j_p e^k_r \sigma^p \sigma^r = \frac{1}{2i} \sum_{p,r} \epsilon_{pr} \sigma^p \sigma^r = \sigma^3 = \chi.$$

The computation that the Hochschild boundary of c vanishes is straightforward and leads to the conditions (2.12) of the lemma. □

This property is especially interesting since it is not known for the so called *gauge* perturbations of the Dirac operator. It is worth to mention that, to the best of our knowledge, it is the first instance (apart from the case of 0-dimensional spectral triples over finite algebras) where the coefficients of the Hochschild cycle have to be taken in a larger A_θ-bimodule $JA_\theta J^* \otimes A_\theta$ rather than in A_θ itself, which usually suffices.
Note that the terms valued in $JA_\theta J^*$ are "spectators" from the point of view of the Hochschild boundary operator and although we obtain a nontrivial condition for the commutation relations between the elements of the coframe E^j_i still it does not restrict it to a completely commutative case. On the other hand, one might think of relaxing the Hochschild boundary condition by requiring, for instance, that $\Phi(bc) = 0$, where $\Phi = \phi \otimes \mathrm{id} \otimes \mathrm{id}$ and ϕ is an arbitrary character on $JA_\theta J^*$.
It is also interesting that the conditions (2.12) for c to be a cycle would automatically enforce that $\det E$ and hence $\sqrt{g}$ can be unambigously defined, so it is then possible to detach $\sqrt{g}$ from the product $\sqrt{|g|}\, R$, and define scalar curvature alone. We leave that issue for further investigations.
Finally, let us comment the link with other operators. We note that the Dirac operator, which we propose is a natural generalization of the operator

$$\sigma^1 \delta_1 + J\rho J^* \sigma^2 \delta_2, \tag{2.13}$$

where ρ is a positive element of A_θ. The class of such operators arose from construction of Dirac operators compatible with connections on noncommutative principal $U(1)$ bundles [10, 11]. We also note that with e^k_j such that:

$$e^1_1 = 1, \;\; e^1_2 = 0, \;\; e^2_2 = \Im(\tau), \;\; e^2_1 = \Re(\tau),$$

we recover D to be just the Dirac operator of the flat metric in the conformal class τ.
Moreover, the family (2.9) include the operators given by eq. (44) in the paper [8] (which however studies the modular, or twisted, spectral triples given by eq (45) therein).
Note that in all those three instances the condition (2.12) is satisfied.

3 The Gauss-Bonnet theorem and curvature on the noncommutative torus

3.1 Symbol calculus on Noncommutative Torus

The symbol calculus defined in [9] and developed further in [8] is easily generalized to the case of the operators defined above.
Let us define a differential operator of order at most n as:

$$P_n = \sum_{i=1,2} \sum_{0 \le j \le n} \sum_{0 \le k \le j} a_{ij} \delta_1^k \delta_2^{j-k},$$

where a_{ij} are in $JA_\theta J^{-1}$.
Let ρ be C^∞ function from $\mathbb{R}^2$ to $J\mathcal{A}_\theta J^*$, which is homogeneous of order n, satisfying certain bounds [9]. For every symbol ρ we define the operator on $\mathcal{H}_\theta$:

$$P_\rho(a) = \frac{1}{(2\pi)^n} \int e^{-is\xi} \rho(\xi) \alpha_s(a) ds dx.$$

where

$$\alpha_s(U^\alpha) = e^{is\cdot\alpha} U^\alpha.$$

The Dirac operator (2.9) could certainly be expressed in this way and its symbol reads:

$$P(D) = \sum_{j,\mu=1}^{2} \left(\sigma^j e_j^\mu \xi_\mu + \frac{1}{2} \sigma^j \delta_\mu(e_j^\mu) \right). \tag{3.1}$$

3.2 The computations

As shown in [9] by pseudodifferential calculations the value $\zeta(0)$ at the origin of the zeta function of the operator D^2 is given by

$$\zeta(0) = -\int \mathfrak{t}(b_2(\xi))\, d\xi,$$

where $b_2(\xi)$ is a symbol of order -4 of the pseudodifferential operator $(D^2+1)^{-1}$. It can be computed by recursion from the symbol $a_2(\xi) + a_1(\xi) + a_0(\xi)$ of D^2 as follows:

$$\begin{aligned} b_2 \quad = \quad & -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_1(b_0)\delta_1(a_1) b_0 + \partial_2(b_0)\delta_2(a_1) b_0 \\ & + \partial_1(b_1)\delta_1(a_2) b_0 + \partial_2(b_1)\delta_2(a_2) b_0 + \tfrac{1}{2}\partial_{11}(b_0)\delta_1^2(a_2) b_0 \\ & \tfrac{1}{2}\partial_{22}(b_0)\delta_2^2(a_2) b_0 + \partial_{12}(b_0)\delta_{12}(a_2) b_0), \end{aligned}$$

where

$$b_1 \quad = \quad -(b_0 a_1 b_0 + \partial_1(b_0)\delta_1(a_2) b_0 + \partial_2(b_0)\delta_2(a_2) b_0),$$

$$b_0 = (a_2 + 1)^{-1}$$

and

$$\partial_1 = \frac{\partial}{\partial \xi_1}, \qquad \partial_2 = \frac{\partial}{\partial \xi_2}.$$

In or case we have

$$a_2(\xi) = \sigma^j \sigma^k e_j^\mu e_k^\nu \xi_\mu \xi_\nu \ ,$$

$$a_1(\xi) = \sigma^j \sigma^k \{ \tfrac{1}{2} e_j^\mu \delta_\nu(e_k^\nu) + e_j^\nu \delta_\nu(e_k^\mu) + \tfrac{1}{2} \delta_\nu(e_j^\nu) e_k^\mu \} \xi_\mu \ ,$$

and

$$a_0(\xi) = \sigma^j \sigma^k \{ \tfrac{1}{4} \delta_\mu(e_j^\mu) \delta_\nu(e_k^\nu) + \tfrac{1}{2} e_j^\mu \delta_\nu(e_k^\nu) \} \ .$$

Due to the noncommutativity the computation of several hundreds of terms occuring in the formula for $\zeta(0)$ is a formidable task, which needs a symbolic calculation assistence and ingeneous manipulations. Even so, it is hardly possible to obtain a final result without any simplifying assumption. That was indeed the case of conformal-type metric in [9].

Instead, we propose to perform perturbative analysis of the resulting terms having assumed that the metric is a slight deviation of the standard equivariant one.

We assume that the components of two-frame can be expanded as

$$e_j^\mu = \delta_j^\mu + \varepsilon h_j^\mu,$$

where δ_j^μ is the Kronecker delta and $h_j^\mu \in JA_\theta J^*$. The initial unperturbed metric is thus flat, with $\tau = i$. This can be assumed without any loss of generality, as we shall explain a posteriori that our result is valid for arbitrary initial constant metric and thus for arbitrary conformal class τ.

For the assumed form of the Dirac-type operator we shall compute the volume form as well as the curvature functional, which in the case of taking its value on the unit of the algebra shall give us the (perturbative expansion) of the Gauss-Bonnet theorem.

3.3 Volume functional and absolute continuity

Let us consider the following functional on the algebra $JA_\theta J^*$:

$$JA_\theta J^* \ni a \mapsto \mathrm{Res}_{s=2} \zeta_a(s),$$

where

$$\zeta_a(s) = \mathrm{Tr}\ a|D|^{-s}.$$

Using the symbolic calculus we can relate the residue [15, 16] to the following expression:

$$\mathrm{Res}_{s=2} \zeta_a(s) = \int_{S^1} \mathfrak{t}(a b_0(\xi)) d\Omega(\xi).$$

In the perturbative expansion one then obtain the relevant volume functionals (as for $a = 1$ one obtains the volume as the leaing term of the spectral action expansion):

$$\mathrm{Res}_{s=2} \zeta_a(s) \sim (2\pi)^2 \mathfrak{t} \left(a \left(1 - \varepsilon (h_1^1 + h_2^2) + \varepsilon^2 \left((h_1^1)^2 + (h_2^2)^2 + \frac{1}{2} \left[h_1^2, h_2^1 \right]_+ + \frac{1}{2} \left[h_1^1, h_2^2 \right]_+ \right) \right) \right) + O(\varepsilon^3), \tag{3.2}$$

Note that the functional $\mathrm{Res}_{s=2} \zeta_b(s)$ taken for $b \in A_\theta$ factorizes, i.e.:

$$\mathrm{Res}_{s=2} \zeta_b(s) \sim (2\pi)^2 \mathfrak{t}(b) \mathfrak{t} \left(\left(1 - \varepsilon (h_1^1 + h_2^2) + \varepsilon^2 \left((h_1^1)^2 + (h_2^2)^2 + \frac{1}{2} \left[h_1^2, h_2^1 \right]_+ + \frac{1}{2} \left[h_1^1, h_2^2 \right]_+ \right) \right) \right) + O(\varepsilon^3), \tag{3.3}$$

For this reason one might wonder whether the absolute continuity condition (see axiom 5 in [6]) for the spectral triple makes sense in an unchanged form.
In particular it might be asked whether the hermitian structure on the trivial module of spinors (which after completion gives the spinor space) should be valued in the algebra A_θ or rather in the opposite algebra $JA_\theta J*$.
Although still the formula provides a reasonable answer, one sees that the notion of volume form and volume measure (as computed perturbatively in (3.2) and (3.3)) becomes slightly different than in the classical case or in the case of flat Dirac operator.

3.4 Scalar curvature

The idea of the definition of scalar curvature appeared first in [7], expressed in terms of the second term of the heat expansion. For our purposes, as the symbols of the Dirac operator and its square are not in the algebra but in its commutant, we need to propose a slightly modified definition.
We have also to take into account, that we consquently use the state $\mathfrak{t}$ corresponding to the 'flat' measure over $\mathbb{T}_\theta$. Thus we search for the analogy not of the classical scalar curvature, but rather of its product with $\sqrt{|g|}$. More precisely, we search for the unique element in $\tilde{R} \in JA_\theta J^*$ such that:

$$\text{Trace}\,(a(D^2)^{-s/2})_{|s=0} = \frac{1}{12\pi}\mathfrak{t}(a\tilde{R}), \qquad \forall a \in JA_\theta J^*.$$

The refinement of this notion (which classically vanishes) is the unique element $\tilde{R}^\gamma \in JA_\theta J^*$ that satisfies:

$$\text{Trace}\,(\gamma a(D^2)^{-s/2})_{|s=0} = \frac{1}{12\pi}\mathfrak{t}(a\tilde{R}^\gamma), \qquad \forall a \in JA_\theta J^*.$$

Observe that these notions certainly make perfect sense, however, one might ask why we do not take into account similar functionals with $a \in A_\theta$ instead. The reason is the fact that the trace factorizes on products of elements from A_θ by the elements from the $JA_\theta J^*$, i.e. for $a \in A_\theta$ and $b \in JA_\theta J^*$:

$$\mathfrak{t}(ab) = \mathfrak{t}(a)\mathfrak{t}(b).$$

Using the calculus of symbols extended to A_θ and $JA_\theta J^*$ one can find that that the curvature functional would then vanish for all a if it vanishes for $a = 1$ (which is the Gauss-Bonnet theorem). We conjecture that this is the case for all admissible Dirac operators on the noncommutative 2-torus. Of course, such situation is very particular for the case considered and might not be generic.
Below we present the results for the curvature and chiral curvature, seen as elements of $JA_\theta J^*$ in this sense. In fact, even in this simple case the computations are much involved, as the expression for $b_2(\xi)$ alone counts 7100 terms (up to ε^2) and its printout takes more than 100 pages. For this reason we present here only the final result, after integration with respect to ξ.

$$
\begin{aligned}
\tilde{R} =& 2\varepsilon \left(+\delta_1\delta_1(h_2^2) + \delta_2\delta_2(h_1^1) - \delta_1\delta_2(h_1^2) - \delta_2\delta_1(h_2^1)\right) \\
&+\varepsilon^2 \Big(\left[h_1^1, \delta_1\delta_2(h_2^1) + (\delta_1)^2(h_2^2) - 2(\delta_2)^2(h_1^1)\right]_+ + \left[h_2^2, \delta_1\delta_2(h_1^2) + (\delta_2)^2(h_1^1) - 2(\delta_1)^2(h_2^2)\right]_+ \\
&\qquad + \left[h_1^2, 2\delta_1\delta_2(h_2^2) + \delta_1\delta_2(h_1^1) - (\delta_2)^2(h_1^2) - (\delta_1)^2(h_2^1) - (\delta_2)^2(h_1^2)\right]_+ \\
&\qquad + \left[h_2^1, 2\delta_1\delta_2(h_1^1) + \delta_1\delta_2(h_2^2) - (\delta_2)^2(h_1^2) - (\delta_1)^2(h_1^2) - (\delta_1)^2(h_2^1)\right]_+ \\
&\qquad + \left[\delta_2(h_1^1), 2\delta_1(h_2^1) + \delta_1(h_1^2)\right]_+ + \left[\delta_1(h_2^2), 2\delta_1(h_1^2) + \delta_2(h_1^2)\right]_+ \\
&\qquad + \left[\delta_1(h_1^1), \delta_1(h_2^2) + \delta_2(h_2^1)\right]_+ + \left[\delta_2(h_2^2), \delta_2(h_1^1) + \delta_1(h_1^2)\right]_+ \\
&\qquad - 2\left[\delta_2(h_1^2), \delta_2(h_2^1) + \delta_1(h_2^1)\right]_+ \\
&\qquad -2\left(\delta_2(h_1^2)\right)^2 - 2\left(\delta_1(h_2^1)\right)^2 - 4\left(\delta_2(h_1^1)\right)^2 - 4\left(\delta_1(h_2^2)\right)^2 \Big) + O(\varepsilon^3)
\end{aligned}
$$

where $[\cdot,\cdot]_+$ denotes anticommutator.

It is easy to see that the linear terms coincide with the ones from the commutative case, whereas in the next order we obtain terms, which are not a straigthforward generalization of the commutative ones. More interesting is the „chiral curvature" $\tilde{R}^\gamma$. In our case, the first order term vanishes (as expected), whereas the second term could be expressed as a sum of commutators:

$$
\begin{aligned}
\tilde{R}^\gamma =& i\varepsilon^2 \big(\left[h_1^1, \delta_2\delta_2(h_2^1 + h_1^2) - \delta_1\delta_2(3h_1^1 + 2h_2^2) + 3\delta_1\delta_1(h_2^1)\right] \\
&+ \left[h_1^2, 3\delta_1\delta_1(h_2^2) - 2\delta_2\delta_2(h_1^1) - 2\delta_1\delta_2(h_1^2) + \delta_1\delta_2(h_2^1)\right] \\
&+ \left[h_2^1, 2\delta_1\delta_1(h_2^2) + 2\delta_1\delta_2(h_2^1) - 3\delta_2\delta_2(h_1^1) - \delta_1\delta_2(h_1^2)\right] \\
&+ \left[h_2^2, \delta_1\delta_2(3h_2^2 + 2h_1^1) - \delta_1\delta_1(h_1^2 + h_2^1) - 3\delta_2\delta_2(h_1^2)\right] \\
&+ 3\left[\delta_1(h_1^1), \delta_1(h_2^1)\right] + \left[\delta_2(h_1^2), \delta_1(h_2^1)\right] - 2\left[\delta_2(h_1^1), \delta_1(h_2^2)\right] - 3\left[\delta_2(h_1^2), \delta_2(h_1^1)\right] \\
&+ 3\left[\delta_2(h_1^2), \delta_2(h_2^2)\right] - 3\left[\delta_2(h_2^1), \delta_2(h_1^1)\right] - \left[\delta_1(h_1^1), \delta_2(h_2^2)\right] - 2\left[\delta_2(h_2^1), \delta_1(h_1^2)\right] \\
&-3\left[\delta_1(h_2^2), \delta_1(h_1^2)\right] + 3\left[\delta_1(h_2^1), \delta_1(h_2^2)\right]\big) + O(\varepsilon^3),
\end{aligned}
$$

so it vanishes in the limiting case of commutative torus.

3.5 The perturbative Gauss-Bonnet theorem

The statement of the Gauss-Bonnet theorem couldbe phrased either as vanishing of the curvature functional at $a = 1$ or, equivalently as:

$$\mathfrak{t}(\tilde{R}) = 0.$$

Using the periodicity of the trace and the fact that trace is invariant with respect to $U(1)$ actions (which translates to $\mathfrak{t}(\delta_i(a)) = 0$ for $i = 1, 2$ and any $a \in JA_\theta J^*$, we obtain that at order ε the statement holds trivially, all terms are derivations of some elements from $JA_\theta J^*$.
At the order ε^2 (we dentote the ε^2 part of $\tilde{R}$ by $\tilde{R}_{\varepsilon^2}$) we need first to use the cyclicity of the trace, and then the Lebniz rule, to reduce (after three consecutive iterations) the expression to the following one:

$$
\begin{aligned}
\mathfrak{t}(\tilde{R})_{\varepsilon^2} =& \mathfrak{t}\big(-4h_1^1\left(\delta_1\delta_2(h_2^1) - \delta_2\delta_1(h_2^1)\right) + h_1^1\left(\delta_1\delta_2(h_1^2) - \delta_2\delta_1(h_1^2)\right) \\
&-3h_1^2\left(\delta_1\delta_2(h_2^2) - \delta_2\delta_1(h_2^2)\right) + h_2^1\left(\delta_2\delta_1(h_2^2) - \delta_1\delta_2(h_2^2)\right)\big) \\
=& 0,
\end{aligned}
$$

(the last equality follows since δ_1 and δ_2 commute).
We stress in the above computations we used the cyclicity and invariance of $\mathfrak{t}$ the properties of the derivations δ_j (like integration by parts rule), but never their explicit form. Therefore, the result we obtained holds for any linear combination (with constant coefficients) of δ_j, in particular for

$$\delta_1 = \delta_1, \quad \delta_2' = (\tau - \overline{\tau})/2i\delta_2,$$

which corresponds to the complex (or equivalently conformal) class on $\mathbb{T}_\theta$ labeled by τ.

4 Conclusions

Although this project has still to be further developed, we have already encountered several new and interesting phenomena. First of all, we observe that the existence of *real* spectral triples is necessary to provide a kind of background geometry. Then one can consistently introduce a class of new Dirac operators, which in turn correspond to new *noncommutative metrics*, extending the so-far considered examples.
Our aim in this paper was to outline the possibilities, which appear to be much broader than originally believed. Still, many of the properties of the introduced objects are to be studied more closely, which we shall adress in future work. We believe that the introduction of families of (curved) metrics might open a new interstin directions in the investigations of noncommutative manifolds as metric spaces.

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